## AP Calculus AB - Optimization Problems

### Types of Optimization Problems:
- Maximize Volume
- Minimize Distance
- Minimize Area
- Minimize Length

### Ex. 1

A manufacturer wants to design an open box having a square base and a surface area of 108 square inches. What dimensions will produce a box with maximum volume?

**STRATEGY:** Write a function and use derivatives to find the maximum value.

Start by writing the PRIMARY EQUATION (the equation for the quantity to be maximized).

**Equation for Volume:** \( V = x^2h \) (\( x \) is the side of the square base, \( h \) is the height of the box).

Find a secondary equation that can be used to write the primary equation with ONE variable.

**SECONDARY EQUATION:** \( S = x^2 + 4(xh) = 108 \) (Surface Area)

Solve the secondary equation for \( h \) and substitute into primary equation.

\[
\begin{align*}
    h &= \frac{108 - x^2}{4x} \\
    V &= x^2 \left( \frac{108 - x^2}{4x} \right) - \frac{x}{4} \left( 108 - x^2 \right)
\end{align*}
\]
Determine "feasible" domain (relative to the real-life problem)

\( V \geq 0, \ x \geq 0 \)

Volume must be +, side of square +

If height = 0, then \( A = x^2 = 108 \), so \( x = \sqrt{108} = 6\sqrt{3} \)

\[ 0 \leq x \leq 6\sqrt{3} \]

Domain

Find critical #s:

\[
\begin{align*}
V' &= 27 - \frac{3}{4} x^2 = 0 \\
\frac{3}{4} x^2 &= 27 \\
x^2 &= 27 \cdot \frac{4}{3} = 36 \\
x &= 6
\end{align*}
\]

Evaluate \( V \) at ALL critical #s AND ENDPOINTS of domain to find maximum.

\[
\begin{align*}
V(0) &= 0 \\
V(6) &= 27(6) - \frac{1}{4}(6^3) = 162 - 54 = 108 \\
V(6\sqrt{3}) &= 27(6\sqrt{3}) - \frac{1}{4}(6\sqrt{3})^3 = 0
\end{align*}
\]

Max Vol. occurs when \( x = 6 \) \( (6 \times 6 \times 3) \)

Example:

Two posts, one 12 feet high and the other 28 feet high, stand 30 feet apart. They are to be secured by 2 wires, attached to a single stake, running from ground level to the top of each post. Where should the stake be placed to use the least amount of wire?

1) Draw a realistic diagram and label all given information and select variables for unknown information

2) Write a PRIMARY EQUATION for the quantity to be minimized.

3) Write a SECONDARY EQUATION that can be used to change the primary equation to a single variable.

4) Determine 'feasible' domain.

5) Find all critical #s

6) Test the primary equation for all critical #s and endpoints.
Use Graphing Utility to verify.

\[ W' = \frac{1}{a} (x^2 + 144)^{\frac{1}{2}} (2x) + \frac{1}{a} (900 - 60x + x^2 + 784)^{\frac{1}{2}} (2x - 60) \]

\[ = \frac{x}{(x^2 + 144)} + \frac{x - 30}{\sqrt{1684 - 60x + x^2}} = 0 \]

\[ \frac{x - 30}{\sqrt{1684 - 60x + x^2}} \Rightarrow \frac{x}{x^2 + 144} \]

\[ (x-30)(x^2+144) = -x(\sqrt{1684 - 60x + x^2}) \]

Square both sides

\[ (x-30)^2(x^2+144) = x^2(1684-60x+x^2) \]

\[ (x-30)^2(x^2+900)(x+144) = 1684x^2 - 60x^3 + x^4 \]

\[ x^4 - 60x^3 + 900x^2 + 144x^2 - 8640x + 129600 = 1684x^2 - 60x^3 + x^4 \]

\[ 0 = 640x^2 + 8640x - 129600 \]

\[ 0 = 320(2x^2 + 27x - 405) \]

\[ = 320(2x + 45)(x - 9) \]

\[ x = \frac{-45}{2} \quad \text{or} \quad x = 9 \]

\[ \text{Not in domain, } \text{critical} \# \]

To find min. on \([0,30]\), test endpoints and critical #

\[ W(0) = 53.037 \]

\[ W(30) = 60.311 \]

\[ \boxed{W(9) = 50} \Rightarrow \text{Minimum amount of wire!} \]

See Example 5, pg. 222 in textbook for a case where the maximum value occurs at the ENDPOINT of the domain (not a critical point).